UDC 62-50

PROBLEMS OF CONFLICT CONTROL

PMM Vol. 39, № 2, 1975, pp. 225-234

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(Received June 27, 1974)

We develop an auxiliary program construction which provides a method for constructing the resolving strategies in a linear encounter-evasion differential game. We ascertain the relations between the various methods for constructing the resolving strategies.

1. We examine the conflict-controlled system described by the equation

$$d\mathbf{x}/dt = A(t)\mathbf{x} + B(t)\mathbf{u} + C(t)\mathbf{v} \tag{1.1}$$

Here $\mathbf{x}$ is the n-dimensional phase vector; $\mathbf{u}$ and $\mathbf{v}$ are the controls of the first and second players, with the constraints

$$\mathbf{u} \in P, \quad \mathbf{v} \in Q \tag{1.2}$$

where P and Q are convex compacta; $A(t)$, $B(t)$ and $C(t)$ are matrices with elements continuous in t. The players'strategies and the motions they generate are defined according to [1, 2].

Let there be given a closed set M_1 in space $\{\mathbf{x}\}$, the initial position $\{t_0, \mathbf{x}_0\}$ and the instant $t = \vartheta$ $(\vartheta > t_0)$. The first player poses the problem of finding the strategy $\mathbf{U}^\circ \div \mathbf{u}^\circ(t, \mathbf{x})$ which ensures the equality

$$\max_{\mathbf{x}[t]} \rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{U}^\circ], M_1) = \min_{\mathbf{U}} \max_{\mathbf{x}[t]} \rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{U}], M_1) \tag{1.3}$$

the second player poses the problem of finding a strategy $\mathbf{V}^\circ \div \mathbf{v}^\circ(t, \mathbf{x})$ such that

$$\min_{\mathbf{x}[t]} \rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{V}^\circ], M_1) = \max_{\mathbf{V}} \min_{\mathbf{x}[t]} \rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{V}], M_1) \tag{1.4}$$

By $\rho(\mathbf{x}[\vartheta], M_1)$ we have denoted the Euclidean distance from point $\mathbf{x}[\vartheta]$ to set M_1.

If the encounter problem is solvable, then there exists a limitedly wide u-stable bridge [1] connecting the initial position with set M_1; conversely, if the evasion problem is solvable, then there exists a v-stable bridge [1] passing through the initial position and bypassing set M_1. On the other hand, if we succeed in constructing a u-stable (v-stable) bridge connecting the initial position with set M_1 (bypassing set M_1), then the strategy U_c (V_c) extremal to this bridge solves the encounter (evasion) problem. Several methods exist for constructing stable bridges (and resolving strategies) on the basis of program constructions [2-6]. In the present paper we develop one such construction and establish the connection between the various constructions.

2. Let us consider the following constructions which are modifications of the constructions in [3]. The arbitrary function $\mathbf{V}(t, \mathbf{u})$ which brings set $\mathbf{V}(t, \mathbf{u}) \subset Q$ into congruence with set $\{t, \mathbf{u}\}$ $(\mathbf{u} \in P)$ will be called the upper subprogram of the second player.

We divide the interval $[t_0, \vartheta]$ into the half-open intervals $[\tau_i, \tau_{i+1})$ $(i = 0, 1, 2, \ldots; \tau_0 = t_0)$ and we define the Euler polygonal line $\mathbf{x}_\Delta^\pi[t] = \mathbf{x}_\Delta^\pi[t, t_0, \mathbf{x}_0, \mathbf{u}[\cdot], \mathbf{V}(\cdot, \mathbf{u})]$

as the absolutely continious solution of the equation

$$\frac{d\mathbf{x}_\Delta{}^\pi[t]}{dt} = A(t)\,\mathbf{x}_\Delta{}^\pi[t] + B(\tau_i)\,\mathbf{u}[\tau_i] + C(\tau_i)\,\mathbf{v}(\tau_i,\ \mathbf{u}[\tau_i]) \tag{2.1}$$
$$(\tau_i \leqslant t \leqslant \tau_{i+1}),\ \ \mathbf{v}(\tau_i,\ \mathbf{u}[\tau_i]) \in \mathbf{V}(\tau_i,\ \mathbf{u}[\tau_i]))$$

with the initial condition $\mathbf{x}_\Delta^\pi[t_0] = \mathbf{x}_0$; as the realizations $\mathbf{u}[t] \in P$ we take piece-wise-constant functions, considering that the points of discontinuity of these functions coincide with τ_i. Every limit of a suitable sequence of Euler polygonal lines $\{\mathbf{x}_\Delta^{\pi(k)}[t,\ t_0,\ \mathbf{x}_0{}^{(k)},\ \mathbf{u}^{(k)}[\cdot],\ \mathbf{V}(\cdot,\ \mathbf{u}^{(k)})]\}$ as $k \to \infty$, $\lim_{k\to\infty} \sup_i(\tau_{i+1}^{(k)} - \tau_i^{(k)}) = 0$, $\lim_{k\to\infty} x_0^{(k)} = x_0$ is called a motion $\mathbf{x}^\pi[t] = \mathbf{x}^\pi[t,\ t_0,\ \mathbf{x}_0,\ \mathbf{V}(\cdot,\ \mathbf{u})]$ of system (1.1). The set of all motions generated by the upper subprogram $\mathbf{V}(t,\ \mathbf{u})$ is a compactum in $C_{[t_0,\ \vartheta]}$.

Every sequence $\{\mathbf{V}^{(i)}(t,\ \mathbf{u})\}$ $(i = 1,2,\ldots)$ is called the second player's upper program $\mathbf{\Pi}(t,\ \mathbf{u})$. Suppose that some upper program $\mathbf{\Pi}(t,\ \mathbf{u}) = \{\mathbf{V}^{(i)}(t,\ \mathbf{u})\}$ of the second player is specified; for the subprogram $\mathbf{V}^{(i)}(t,\ \mathbf{u})$ $(i = 1,\ 2,\ldots)$ we construct the set of all motions generated by them. From each set we take arbitrarily one motion and make up the sequence

$$\mathbf{x}^{\pi(1)}[t],\ \mathbf{x}^{\pi(2)}[t],\ \ldots,\ \mathbf{x}^{\pi(k)}[t],\ldots \tag{2.2}$$

where $\mathbf{x}^{\pi(j)}[t]$ is a motion from the set generated by $\mathbf{V}^{(j)}(t,\ \mathbf{u})$. From sequence (2.2) of uniformly bounded and equicontinuous functions we can select a converging subsequence. The limits of all possible subsequences constructed in such a manner form a sheaf $\Gamma(\mathbf{\Pi})$. The motions comprising sheaf $\Gamma(\mathbf{\Pi})$ are denoted by $\mathbf{x}(t,\ \Gamma(\mathbf{\Pi}))$. It can be shown that the sheaf $\Gamma(\Pi)$ is a compactum in $C_{[t_0,\vartheta]}$ and is a v-stable set, i.e. for any position $\{t_*,\ \mathbf{x}_*\} \in \Gamma(\mathbf{\Pi})$, for any $t^*\ (t_* < t^* \leqslant \vartheta)$ and for an arbitrary constant $\mathbf{u}^* \in P$ we can find at least one solution of the contingent differential equation

$$d\mathbf{x}\ /\ dt \in A\ (t)\ \mathbf{x} + B\ (t)\ \mathbf{u}^* + C\ (t)\ Q \tag{2.3}$$

satisfying the condition $\{t^*,\ \mathbf{x}(t^*)\} \in \Gamma(\Pi)$.

The proof is based on the following two fundamental facts. First, there exists a sequence of motions $\{\mathbf{x}^{\pi(i)}[t]\}$ of system (1.1), generated by the upper subprograms $\mathbf{V}^{(i)}(t,\mathbf{u})$, converging uniformly to some motion from the sheaf $\Gamma(\Pi)$, passing through the point $\{t_*,\mathbf{x}_*\}$ and such that the vector sequence $\{\mathbf{x}^{\pi(i)}[t_*]\}$ converges to $\mathbf{x}^*$. Second, the uniform limit of the converging subsequence of Euler polygonal lines of system (1.1), constructed for $\mathbf{u}[t] = \mathbf{u}^*,\ t \in [t_*,\ t^*]$, and for an arbitrary upper subprogram of the second player, is a solution of the contingent equation (2.3).

We assume that set M_1 is convex and finite. we denote an R-neighborhood of M_1 by $M_1{}^R$. We define a closed set $M_2 = \{\mathbf{x} : \mathbf{x} \in E_n \setminus M_1{}^R\}$, where we take the number R to be so large that the distance $\rho(\mathbf{x}[\vartheta],\ M_2)$ from the point $\mathbf{x}[\vartheta]$ to set M_2 is greater than zero for any motion $\mathbf{x}[t]$ of system (1.1). Then for any point $\mathbf{p} \in \{\mathbf{x} : \mathbf{x} \in M_1{}^R \setminus M_1\}$ there exists a unique straight line passing through point $\mathbf{p}$, such that $\rho(\mathbf{p},\ M_1) + \rho(\mathbf{p},\ M_2) = R$.

Obviously, among the upper programs we can find a program $\mathbf{\Pi}_0(t,\ \mathbf{u}) = \{\mathbf{V}_0{}^{(i)}(t,\ \mathbf{u})\}$ such that

$$\lim_{i\to\infty} \max_{\mathbf{x}^{\pi(i)}[t]} \rho(\mathbf{x}^{\pi(i)}[\vartheta], M_2) = \inf_{\mathbf{V}(t,\ u)} \max_{\mathbf{x}^{\pi}[t]} \rho(\mathbf{x}^{\pi}[\vartheta], M_2) \tag{2.4}$$

where $\mathbf{x}^{\pi(i)}[t]$ denotes the motion corresponding to the subprogram $\mathbf{V}_0{}^{(i)}(t,\ \mathbf{u})$. We can also verify the validity of the equality

$$\min_{\Pi(t,u)} \max_{x(t,\Gamma(\Pi))} \rho(x(\vartheta, \Gamma(\Pi)), M_2) = \inf_{V(t,u)} \max_{x^{\pi}[t]} \rho(x^{\pi}[\vartheta], M_2) \quad (2.5)$$

where $\Gamma(\Pi_0)$ is the sheaf corresponding to the upper program $\Pi_0(t, u)$.

Problem 2.1. Find the minimizing upper program $\Pi_0(t, u)$ and in the sheaf $\{x(t, \Gamma(\Pi_0))\}$ generated by it find a curve $x^{\circ}(t, \Gamma(\Pi_0))$ such that

$$\rho(x^{\circ}(\vartheta, \Gamma(\Pi_0)), M_2) = \min_{\Pi(t,u)} \max_{x(t,\Gamma(\Pi))} \rho(x(\vartheta), \Gamma(\Pi)), M_2) = \quad (2.6)$$
$$\varepsilon_1(\vartheta, t_0, x_0, M_2)$$

Problem 2.2. Find the maximizing upper program $\Pi^{\circ}(t, u)$ and in the sheaf $\{x(t, \Gamma(\Pi^{\circ}))\}$ find a curve $x^{\circ}(t, \Gamma(\Pi^{\circ}))$ such that

$$\rho(x^{\circ}(\vartheta, \Gamma(\Pi^{\circ})), M_1) = \max_{\Pi(t,u)} \min_{x(t,\Gamma(\Pi))} \rho(x(\vartheta, \Gamma(\Pi)), M_1) = \quad (2.7)$$
$$\varepsilon_2(\vartheta, t_0, x_0, M_1)$$

From the preceding discussion it follows that at least one solution of Problem 2.1 exists.

The following statement is valid. If program Π_0 and curve $x^{\circ}(t, \Gamma(\Pi_0))$ provide the solution of Problem 2.1, then they also provide the solution of Problem 2.2. Conversely, if the pair $\{\Pi^{\circ}, x^{\circ}(t, \Gamma(\Pi^{\circ}))\}$ provides the solution of Problem 2.2 and if here $\varepsilon_2^{(1)} > 0$, then this pair provides the solution of Problem 2.1 as well and

$$\varepsilon_1^{(2)} + \varepsilon_2^{(1)} = R \quad (2.8)$$

Here and below we introduce the notation $\varepsilon_j^{(i)} = \varepsilon_j(\vartheta, t_0, x_0, M_i)$, $i = 1, 2$; $j = 1, 2, \ldots$ The solutions of Problems 2.1 and 2.2 may not coincide if set M_1 is not convex.

The sheaf $\Gamma(\Pi_0)$, where Π_0 provides the solution of Problem 2.1 is a v-stable set. Consequently [1], the strategy $V_1 \div v_1(t, x)$, extremal to this set, ensures the fulfillment of the inequality

$$\rho(x[\vartheta, t_0, x_0, V_1], M_2) \leqslant \varepsilon_1^{(2)} \quad (2.9)$$

for every action of the first player.

3. Let us now construct the stable sets using a program extremal construction. By program controls we mean any measurable functions $u(t)$ and $v(t)$ $(t_0 \leqslant t \leqslant \vartheta)$ satisfying inclusions (1.2). A program motion $x(t, t_0, x_0, u(\cdot), v(\cdot))$ is defined as an absolutely continuous solution of the equation

$$dx / dt = A(t)x + B(t)u(t) + C(t)v(t), \quad x(t_0) = x_0 \quad (3.1)$$

Problem 3.1. Find maximin program controls $u^{\circ}(t)$, $v^{\circ}(t)$ satisfying the condition

$$\rho(x(\vartheta, t_0, x_0, u^{\circ}(\cdot), v^{\circ}(\cdot)), M_1) = \max_{v(t)} \min_{u(t)} \rho(x(\vartheta, t_0, x_0, \quad (3.2)$$
$$u(\cdot), v(\cdot)), M_1) = \varepsilon_3^{(1)}$$

Problem 3.1 has at least one solution for every choice of initial position (see [2], for example). Let $S(t_0, x_0)$ denote the set of all vectors $s = s(t_0)$ which are solutions of the Cauchy problem

$$ds/dt = -A'(t)s, \quad s(\vartheta) = \left[\frac{\partial\rho(\mathbf{x}, M_1)}{\partial\mathbf{x}}\right]_{\{\vartheta, \mathbf{x}^\circ(\vartheta)\}} \tag{3.3}$$

and which correspond to all possible optimal solutions $\mathbf{x}^\circ(t)$ of Problem 3.1.

Condition 3.1. For arbitrary initial position $\{t_0, \mathbf{x}_0\}$, with any choice of vector $\mathbf{v}^* \in Q$ we can find a vector $\mathbf{u}^* \in P$ such that for all $\mathbf{s} \in S(t_0, x_0)$

$$\mathbf{s}'(B(t)\mathbf{u}^* + C(t)\mathbf{v}^*) \leqslant \max_{\mathbf{v}\in Q} \min_{\mathbf{u}\in P} \mathbf{s}'(B(t)\mathbf{u} + C(t)\mathbf{v}) \tag{3.4}$$

Problem 3.1 is said to be regular if Condition 3.1 is satisfied. If Problem 3.1 is regular, the set

$$W_u = \{t, \mathbf{x} : t_0 \leqslant t \leqslant \vartheta, \ \varepsilon_3^{(1)} \leqslant c\}$$

is a u-stable bridge [6] and, consequently, the strategy $\mathbf{U}_1 \div \mathbf{u}_1(t, x)$ extremal to set W_u, ensures the fulfillment of the inequality

$$\rho(\mathbf{x}[\vartheta, t_0, x_0, \mathbf{U}_1], M_1) \leqslant \varepsilon_3^{(1)} \tag{3.5}$$

The equality

$$\varepsilon_2^{(1)} = \varepsilon_3^{(1)} \tag{3.6}$$

is valid under the assumptions made above.

In fact, from inequalities (2.9) and (3.5), with due regard to relation (2.8), it follows that

$$\varepsilon_3^{(1)} \geqslant \varepsilon_2^{(1)} \tag{3.7}$$

Let us show that the reverse inequality

$$\varepsilon_3^{(1)} \leqslant \varepsilon_2^{(1)} \tag{3.8}$$

holds. Suppose that the measurable functions $\mathbf{u}^\circ(t)$ and $\mathbf{v}^\circ(t)$ provide a solution of Problem 3.1. The corresponding motion $\mathbf{x}^\circ(t) = \mathbf{x}(t, t_0, x_0, \mathbf{u}^\circ(\cdot), \mathbf{v}^\circ(\cdot))$ can be represented by the Cauchy formula

$$\mathbf{x}^\circ(t) = X(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} X(t, \tau)B(\tau)\mathbf{u}^\circ(\tau)\,d\tau + \int_{t_0}^{t} X(t, \tau)C(\tau)\mathbf{v}^\circ(\tau)\,d\tau \tag{3.9}$$

where $X(t, t_0)$ is the fundamental matrix of solutions of the equation $\mathbf{x}^{\cdot} = A(t)\mathbf{x}$. By Luzin's theorem [7], for any $\delta > 0$ the measurable function $\mathbf{v}^\circ(t)$ can be approximated on $[t_0, \vartheta]$ by a continuous function $\mathbf{v}^*(t)$ so that for any arbitrary $\mathbf{u}_*(t) \in P$

$$\rho(\mathbf{x}(\vartheta, t_0, \mathbf{x}_0, \mathbf{u}_*(\cdot), \mathbf{v}^*(\cdot)), M_1) \geqslant \rho(\mathbf{x}(\vartheta\ t_0, \mathbf{x}_0, \mathbf{u}_*(\cdot), \mathbf{v}^\circ(\cdot)), M_1) - \delta \geqslant \varepsilon_3^{(1)} - \delta \tag{3.10}$$

We choose the function $\mathbf{v}^*(t)$ as an upper subprogram $\mathbf{V}(t, u) = \mathbf{v}^*(t)$ and among the motions generated by it we find the one closest to set M_1 at the instant $t = \vartheta$. This motion is the limit of a sequence of Euler polygonal lines corresponding to the sequence of piecewise-constant realizations $\{\mathbf{u}^{(k)}[t]\}$. Having chosen from the sequence $\{\mathbf{u}^{(k)}[t]\}$ a weakly converging subsequence $\{\mathbf{u}^{(k_j)}[t]\}$, we denote its weak limit by $\mathbf{u}^*[t]$. The limit motion $\mathbf{x}^\pi[t] = \mathbf{x}^\pi[t, t_0, \mathbf{x}_0, \mathbf{v}^*(\cdot)]$ is represented by the formula

$$\mathbf{x}^\pi[t] = X(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} X(t, \tau)B(\tau)\mathbf{u}^*[\tau]\,d\tau + \int_{t_0}^{t} X(t, \tau)C(\tau)\mathbf{v}^*(\tau)\,d\tau \tag{3.11}$$

where the second integral is a Riemann integral. Thus, the program motion $\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u}^*(\cdot), \mathbf{v}^*(\cdot))$ can be treated as a motion generated by the upper program $\mathbf{V}(t, \mathbf{u}) = \mathbf{v}^*(t)$

and by the sequence of realizations $\{\mathbf{u}^{(k_j)}[t]\}$ converging weakly to $\mathbf{u}^*(t)$. Therefore,

$$\rho(\mathbf{x}(\vartheta, t_0, \mathbf{x}_0, \mathbf{u}^*(\cdot), \mathbf{v}^*(\cdot)), M_1) = \min_{\mathbf{x}^\pi[t]} \rho(\mathbf{x}^\pi[\vartheta, t_0, \mathbf{x}_0, \mathbf{v}^*(\cdot)], M_1) \leqslant \varepsilon_2^{(1)} \quad (3.12)$$

Comparing (3.10) and (3.12) and taking into account that δ can be chosen arbitrarily small, we arrive at inequality (3.8). Now, (3.6) follows from (3.7) and (3.8).

4. Let us discuss the connection of the constructions from Sect. 2 with the constructions of a priori stable bridges in the form of integral manifolds generated by contingent equations [2, 6], which correspond to the direct method [4, 5] in the formalization of [2].

We define the set

$$G(t) = \bigcap_{\mathbf{u}\in P} (B(t)\mathbf{u} + C(t)Q) \quad (4.1)$$

and assume that it is nonempty. We consider the contingent equation

$$\mathbf{x}^\cdot \in A(t)\mathbf{x} + G(t) \quad (4.2)$$

All solutions $\mathbf{x} = \varphi(t)$ of Eq. (4.2) possess the property that the path $\{t, \mathbf{x} = \varphi(t)\}$ is v-stable. The collection of all solutions of Eq. (4.2) forms an integral manifold Φ which is a compactum.

Problem 4.1. Find the motion $\mathbf{x}_\varphi^\circ(t) = \mathbf{x}_\varphi^\circ(t, t_0, \mathbf{x}_0)$ from manifold Φ, satisfying the condition

$$\rho(\mathbf{x}_\varphi^\circ(\vartheta, t_0, \mathbf{x}_0), M_2) = \min_{\mathbf{x}_\varphi\in\Phi} \rho(\mathbf{x}_\varphi(\vartheta, t_0, \mathbf{x}_0), M_2) = \varepsilon_4^{(2)} \quad (4.3)$$

Let us show that the equality

$$\varepsilon_4^{(2)} = \varepsilon_1^{(2)} \quad (4.4)$$

is satisfied under a certain additional condition. Let the minimum in (4.3) be reached on a function $\mathbf{x}_\varphi^\circ(t) \in \Phi$ satisfying the equation

$$\mathbf{x}_\varphi^{\cdot\circ} = A(t)\mathbf{x}_\varphi^\circ + \mathbf{g}^\circ(t) \quad (4.5)$$

where $\mathbf{g}^\circ(t) \in G(t)$. It follows from the definition of set $G(t)$ that $\mathbf{g}^\circ(t) \in B(t)\mathbf{u}^*[t] + C(t)Q$ for any $\mathbf{u}^*[t] \in P$. In accordance with Luzin's theorem [7] we approximate the measurable function $\mathbf{g}^\circ(t)$ by the continuous function $\mathbf{g}^*(t)$ which on $[t_0, \vartheta]$ differs from $\mathbf{g}^\circ(t)$ in a set of reasonably small measure.

We specify a function $\mathbf{v}(\tau_i, \mathbf{u}_i)$ for $t_0 \leqslant \tau_i \leqslant \vartheta$, $\mathbf{u}_i \in P$ ($\mathbf{u}_i$ is a constant), by the relation

$$\mathbf{g}^*(\tau_i) = B(\tau_i)\mathbf{u}_i + C(\tau_i)\mathbf{v}(\tau_i, \mathbf{u}_i) \quad (4.6)$$

It is known a priori that at least one solution of system (4.6) exists. We denote this solution by $\mathbf{v}^*(\tau_i, \mathbf{u}_i)$ and write the equation

$$\frac{d\mathbf{x}_\Delta[t]}{dt} = A(t)\mathbf{x}_\Delta[t] + B(\tau_i)\mathbf{u}[\tau_i] + C(\tau_i)\mathbf{v}^*(\tau_i, \mathbf{u}[\tau_i]) \quad (4.7)$$

where $\mathbf{u}[t]$ is a piecewise-constant function whose points of discontinuity coincide with τ_i. A solution $\mathbf{x}_\Delta[t]$ of Eq. (4.7) can be interpreted as an Euler polygonal line generated by the upper subprogram $\mathbf{V}(t, \mathbf{u}) = \mathbf{v}^*(t, \mathbf{u})$. Under a suitable choice of $\mathbf{v}^*(t, \mathbf{u})$ the motion generated by this upper subprogram differs arbitrarily little from the motion $\mathbf{x}_\varphi^\circ(t)$. Therefore,

$$\varepsilon_1^{(2)} \leqslant \varepsilon_4^{(2)} \tag{4.8}$$

We say that Problem 4.1 is regular if the function

$$\varkappa_1(\mathbf{l}) = -\max_{\mathbf{u}\in P} \mathbf{l}' B(t)\mathbf{u} - \min_{\mathbf{v}\in Q} \mathbf{l}' C(t)\mathbf{v} \tag{4.9}$$

is convex with respect to $\mathbf{l}$. In this case the first player has at his disposal the strategy $\mathbf{U}_2 \div \mathbf{u}_2(t, \mathbf{x})$, ensuring him the result [6]

$$\rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{U}_2], M_2) \geqslant \varepsilon_4^{(2)} \tag{4.10}$$

for every action of the second player. Comparing (2.9) and (4.10), we conclude that

$$\varepsilon_4^{(2)} \leqslant \varepsilon_1^{(2)} \tag{4.11}$$

Now, (4.4) follows from (4.11) and (4.8).

We define the set

$$H(t) = \bigcap_{\mathbf{v}\in Q} (B(t)P + C(t)\mathbf{v}) \tag{4.12}$$

and assume that it is nonempty. Then the solutions $\mathbf{x} = \psi(t)$ of the equation

$$\mathbf{x}^{\cdot} \in A(t)\mathbf{x} + H(t) \tag{4.13}$$

possess the property that every path $\{t, \mathbf{x} = \psi(t)\}$ $(t_0 \leqslant t \leqslant \vartheta)$ is u-stable. The collection of all solutions of Eq. (4.13) forms an integral manifold Ψ which is a compactum.

Problem 4.2. Find the motion $\mathbf{x}_\psi^0(t) = \mathbf{x}_\psi^0(t, t_0, \mathbf{x}_0)$ from manifold Ψ, satisfying the condition

$$\rho(\mathbf{x}_\psi^0(\vartheta, t_0, \mathbf{x}_0), M_1) = \min_{\mathbf{x}_\psi\in\Psi} \rho(\mathbf{x}(\vartheta, t_0, \mathbf{x}_0), M_1) = \varepsilon_5^{(1)} \tag{4.14}$$

From the u-stability of path $\{t, \mathbf{x}_\psi^0(t)\}$ it follows that a strategy $\mathbf{U}_3 \div \mathbf{u}_3(t, \mathbf{x})$, exists, ensuring the fulfillment of the inequality

$$\rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{U}_3], M_1) \leqslant \varepsilon_5^{(1)} \tag{4.15}$$

for every action of the second player. Comparing (4.15) with (2.9) and (3.8), we obtain

$$\varepsilon_5^{(1)} \geqslant \varepsilon_2^{(1)} \geqslant \varepsilon_3^{(1)} \tag{4.16}$$

We say that Problem 4.2 is regular if the function

$$\varkappa_2(\mathbf{l}) = -\min_{\mathbf{u}\in P} \mathbf{l}' B(t)\mathbf{u} - \max_{\mathbf{v}\in Q} \mathbf{l}' C(t)\mathbf{v} \tag{4.17}$$

is convex with respect to $\mathbf{l}$. In this case we can find a strategy $\mathbf{V}_2 \div \mathbf{v}_2(t, \mathbf{x})$, ensuring the result

$$\rho(\mathbf{x}[\vartheta, t_0, \mathbf{x}_0, \mathbf{V}_2], M_1) \geqslant \varepsilon_5^{(1)}$$

for every action of the first player. Let us show that if Problem 4.2 is regular, the equality

$$\varepsilon_3^{(1)} = \varepsilon_5^{(1)} \tag{4.18}$$

is fulfilled. In fact, let us write out the explicit expressions for the quantities $\varepsilon_3^{(1)}$ and $\varepsilon_5^{(1)}$ (see [8], for example)

$$\varepsilon_3^{(1)} = \max_{\|\mathbf{l}\|=1} \Big(\int_{t_0}^{\vartheta} \max_{\mathbf{v}\in Q} \min_{\mathbf{u}\in P} \mathbf{l}'X(\vartheta,\tau)[B(\tau)\mathbf{u} + C(\tau)\mathbf{v}]\,d\tau + \mathbf{l}'X(\vartheta,t_0) - \rho_{M_1}(\mathbf{l})\Big) \quad \text{for} \quad \varepsilon_3^{(1)} > 0 \tag{4.19}$$

$$\varepsilon_5^{(1)} = \max_{\|\mathbf{l}\|=1} \Big(\int_{t_0}^{\vartheta} \min_{\mathbf{h}^*\in H^*} l'X(\vartheta,t)\,\mathbf{h}^*(t)\,dt + l'X(\vartheta,t_0)\mathbf{x}_0 - \rho_{M_1}(\mathbf{l})\Big) \quad \text{for} \quad \varepsilon_5^{(1)} > 0$$

$$H^*(t) = \{\mathbf{h}^* : \mathbf{s}'\mathbf{h}^* \geqslant \max_{\mathbf{v}\in Q}\min_{\mathbf{u}\in P} \mathbf{s}'[B(t)\mathbf{u} + C(t)\mathbf{v}] = -\varkappa_2(\mathbf{s})\}$$

Here $X(t, t_0)$ is the fundamental matrix of solutions of the equation $\mathbf{x}^{\cdot} = A(t)\mathbf{x}$; $\rho_{M_1}(\mathbf{l})$ is the support function of set $-M_1$. The function $\varkappa_2(\mathbf{s})$ is convex by definition; therefore, for any $\mathbf{s}$ we can find a vector $\mathbf{h}^*(\mathbf{s})$ such that the equality [9]

$$\mathbf{s}'\mathbf{h}^*(\mathbf{s}) = \max_{\mathbf{v}\in Q} \min_{\mathbf{u}\in P} \mathbf{s}'[B(t)\mathbf{u} + C(t)\mathbf{v}] \tag{4.20}$$

is satisfied. Now, (4.18) follows from (4.19) and (4.20). allowing for (4.16), in this case we also have $\varepsilon_5^{(1)} = \varepsilon_2^{(1)}$.

5. Let us discuss the connection between the regularity of Problem 4.1 and the regularity of Problem 3.1. Let the set $G(t)$ of (4.1) be nonempty and let the function $\varkappa_1(\mathbf{l})$ be convex with respect to $\mathbf{l}$. This implies that for every choice of the vector $\mathbf{v} \in Q$ the set

$$F_u(t, \mathbf{v}) = B(t)P + C(t)\mathbf{v} \tag{5.1}$$

intersects $G(t)$. Vector $\mathbf{g}(t)$ is contained in $G(t)$ if and only if the inequality

$$\mathbf{l}'\mathbf{g} \geqslant \max_{\mathbf{u}\in P} \min_{\mathbf{v}\in Q} \mathbf{l}'(B(t)\mathbf{u} + C(t)\mathbf{v}) \tag{5.2}$$

is valid for every choice of vector $\mathbf{l}$ But this means that Condition 3.1 is fulfilled. In fact, for any vector $\mathbf{v}^* \in Q$, in the set $F_u(t, v^*)$ we can find a vector $g^* = B(t)\mathbf{u}^* + C(t)\mathbf{v}^*$ satisfying the inequality

$$\mathbf{l}'(B(t)\mathbf{u}^* + C(t)\mathbf{v}^*) \geqslant \max_{\mathbf{u}\in P} \min_{\mathbf{v}\in Q} \mathbf{l}'(B(t)\mathbf{u} + C(t)\mathbf{v}) \tag{5.3}$$

for all $\mathbf{l}$. In particular, if $\mathbf{l} = -\mathbf{s}(t_0)$, where $\mathbf{s}(t_0) \in S(t_0, \mathbf{x}_0)$ (see (3.3)), then inequality (3.4) follows from (5.3), i.e. Condition 3.1 is fulfilled. Thus, if Problem 4.1 is regular, then Problem 3.1 is regular.

We formulate the main conclusions in the following theorem.

Theorem. Let the encounter-evasion differential game (1.4), (1.5) be given ; let the set M_1 be closed and convex; let $\{t_0, \mathbf{x}_0\}$ be the initial position. Then:

1°. If Problem 3.1 is regular, the strategy pair $\{\mathbf{U}_1, \mathbf{V}_1\}$ defined by conditions (3.5) and (2.9) forms a saddle point of the game and $\varepsilon_3^{(1)} = \varepsilon_2^{(1)}$.

2°. If Problem 4.1 is regular, the strategy pair $\{\mathbf{U}_2, \mathbf{V}_1\}$ defined by conditions (4.10) and (2.9) forms a saddle point of the game and $R - \varepsilon_4^{(2)} = \varepsilon_2^{(1)}$.

3°. If Problem 4.1 is regular, Problem 3.1 is regular as well and $R - \varepsilon_4^{(2)} = \varepsilon_3^{(1)} = \varepsilon_2^{(1)} = R - \varepsilon_1^{(2)}$.

4°. If Problem 4.2 is regular, $\varepsilon_5^{(1)} = \varepsilon_2^{(1)} = \varepsilon_3^{(1)}$.

6. Let us present an illustrative example for the system

$$y_1^{\cdot} = y_3, \quad y_2^{\cdot} = y_4, \quad y_3^{\cdot} = u_1, \quad y_4^{\cdot} = u_2, \quad z_1^{\cdot} = v_1, \quad z_2^{\cdot} = v_2 \tag{6.1}$$

$$\| \mathbf{u} \| = (u_1^2 + u_2^2)^{1/2} \leqslant \mu, \qquad \| \mathbf{v} \| = (v_1^2 + v_2^2)^{1/2} \leqslant \nu$$

analyzed for another purpose and in another way in [10]. The initial instant t_0 and the game termination instant ϑ are specified, and $t_0 < \vartheta - \nu/\mu$. Let the first player minimize the distance from the set $M = [\{\mathbf{y}, \mathbf{z}\}: (y_1 - z_1)^2 + (y_2 - z_2)^2 \leqslant R^2]$. We have

$$x_k^{\cdot} = u_k(\vartheta - t) - v_k, \qquad k = 1, 2 \tag{6.2}$$

in the new variables $x_k = y_k - z_k + y_{k+2}(\vartheta - t)$, $k = 1, 2$. Here $M = \{\mathbf{x} : \| \mathbf{x} \| \leqslant R\}$. Then according to (3.2) and (4.19)

$$\varepsilon_3 = \max_{\| \mathbf{l} \| = 1} [\mathbf{l}'\mathbf{x}_* - r(t_*, \vartheta, \mathbf{l})], \quad \mathbf{l} = \{l_1, l_2\}' \tag{6.3}$$

$$r(t_*, \vartheta, \mathbf{l}) = (l_1^2 + l_2^2)^{1/2} ({}^1/_2\mu (\vartheta - t_*)^2 - \nu(\vartheta - t_*) + R)$$

For Problem 3.1 to be regular it is sufficient that the function $r(t_*, \vartheta, \mathbf{l})$ be convex in $\mathbf{l}$. This yields the condition

$$R \geqslant \max_{t_* \in [t_0, \vartheta]} [\nu(\vartheta - t_*) - {}^1/_2\mu(\vartheta - t_*)^2] = {}^1/_2\nu^2/\mu$$

Thus, Problem 3.1 is regular when $R = {}^1/_2\nu^2/\mu$. Consequently, a strategy exists for the first player, ensuring encounter with the set M (for $R = {}^1/_2\nu^2/\mu$), provided that the initial position $\{t_0, \mathbf{x}_0\}$ satisfies the condition $\varepsilon_3 \leqslant 0$, or

$$\| \mathbf{x}_0 \| \leqslant {}^1/_2\mu (\vartheta - t_0 - \nu/\mu)^2 \tag{6.4}$$

On the other hand, the set $G(t)$ of (4.1) is empty here for $t_0 \leqslant t \leqslant \vartheta - \nu/\mu$ and, consequently, the statement of problem 4.1 becomes meaningless.

The sets $H(t)$ of (4.12) are circles of radius $\mu(\vartheta - t - \nu/\mu)$ for $t_0 \leqslant t \leqslant \vartheta - \nu/\mu$ and are empty for $\vartheta - \nu/\mu < t \leqslant \vartheta$. Consider the auxiliary system

$$x_k^{\cdot *} = u_k(\vartheta - t) - v_k + f(t) u_k, \quad k = 1, 2 \tag{6.5}$$

$$f(t) = \begin{cases} 0, & t_0 \leqslant t < \vartheta - \nu/\mu \\ t - \vartheta + \nu/\mu, & \vartheta - \nu/\mu \leqslant t \leqslant \vartheta \end{cases}$$

For (6.5) the sets $H^*(t)$ of (4.12) are circles of radius $\mu(\vartheta - t - \nu/\mu)$ for $t_0 \leqslant t \leqslant \vartheta - \nu/\mu$, while each of them consists of the single point $\{0, 0\}$ for $\vartheta - \nu/\mu < t \leqslant \vartheta$.

Among the solutions of the equation

$$\mathbf{x}^{\cdot *} = \mathbf{h}^*(t), \quad \mathbf{h}^*(t) \in H^*(t) \tag{6.6}$$

with initial conditions (6.4) we can find a solution $\mathbf{x}^* = \mathbf{w}^*(t)$ such that $\mathbf{w}^*(\vartheta) = 0$. However, if the initial conditions do not satisfy (6.4), then no such solution exists for Eq. (6.6). The path $\{t, \mathbf{w}^*(t)\}$ $(t_0 \leqslant t \leqslant \vartheta)$ is u-stable by construction ; therefore, a strategy exists for the first player, leading the motion of system (6.5) from the initial position (6.4) to the origin. The estimate $\| \mathbf{x}(t) - \mathbf{x}^*(t) \| \leqslant {}^1/_2\nu^2/\mu$ holds, where $\mathbf{x}(t)$, $\mathbf{x}^*(t)$ are the solutions of systems (6.2), (6.5), respectively, with one and the same initial conditions, Hence it follows that a strategy exists for the first player, leading the motion of system (6.2) with initial conditions (6.4) onto the set M $(R = {}^1/_2\nu^2/\mu)$ at the instant ϑ i.e. we have the same conclusion as we obtained above by using the extremal construction.

The author thanks N.N.Krasovskii for posing the problem and for valuable advice.

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Translated by N.H.C.

UDC 518:519.3:62.50

METHOD FOR THE APPROXIMATE SOLUTION OF THE BELLMAN EQUATION FOR PROBLEMS OF OPTIMAL CONTROL OF SYSTEMS SUBJECT TO RANDOM PERTURBATIONS

PMM Vol.39, №2, 1975, pp.235-245

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(Received January 15, 1974)

We propose a method for the approximate soluti n of the Bellman equation for problems of optimal control of the final state o system containing Gaussian white noise of small intensity. We examine the case when the solutions of the deterministic Bellman equation, corresponding to a noisefree system, have discontinuities of the first kind in their own values or in the values of their derivatives. We have found the necessary and sufficient conditions for the synthesis of optimal control of a system additively containing Gaussian white noise to coincide with the corresponding synthesis for the deterministic problem. We prove estimates on the error in the method and we cite examples. Earlier the author had examined an analogous method for a restricted class of optimal control problems [1]. Certain methods for the approximate solution of the Bellman equation were studied in